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THE "RULE OF QUANTITY" IN SPANISH ALGEBRAS OF THE 16TH CENTURY. POSSIBLE SOURCES

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The "rule of quantity" in Spanish Algebras of the 16th century. Possible sources.

Summary: *The paper analyses the treatment of the systems of equations by the more notable authors of the Iberian Peninsula in the 16th Century. The works studied allows us to show the evolution of these systems. It was very important in the process of algebraization of the mathematics. Some hypotheses about the sources that inspired the authors studied have been proposed.*

Key words: *Algebra, Systems of equations, Iberian Peninsula, 16th Century.*

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Introduction

In 16th century Europe, mathematics underwent deep changes whose diffusion was favoured by the invention in the previous century of printing, which completely changed the way that culture was transmitted. One of the main changes was the progressive development of algebra from practical arithmetic, in which the symbolism played a relevant role. It is important to remark that at that time algebra was entirely rhetorical: equations and operations were expressed in words, written out in full.

Between the time of Leonardo de Pisa[1] (1170-1250) and the 16th century, scholars developed abbreviations for many of the words, such as writing *m* for "minus", but they did not introduce any kind of standardized notation.

The best known and probably the most influential Renaissance algebra, was the *Summa de Arithmetica, Geometria, Proportioni and Proportionalità* (1494) by Luca Pacioli (1445-1514). It was written in the vernacular and was a compilation of unpublished works that the author had composed earlier, as well as of general mathematical knowledge at the time. It was, together with Euclid's *Elements*, a reference work for Iberian authors in the 16th century.

The *rule of quantity* or the *rule of the second quantity* are the expressions used in the first treatises on algebra to refer to a procedure of solving problems in which more than one unknown were involved. The first appearance of the second unknown in Western culture was probably around 1373 in the *Trattato di Fioretti* by Antonio de Mazzinghi.[2] The use of more than one unknown would lead to the solution of simultaneous linear equations, whose discussion represented a big step forward in the process of algebraization of mathematics.

Kloyda[3] gave a bibliography of primary printed sources from 1550 to 1660, which contained what nowadays are known as simple and quadratic equations, in order to contribute to a knowledge of mathematical progress in the 16th and 17th centuries. Referring to this bibliography, Grcar[4] points out that in only 4 of the 107 texts mentioned were simultaneous linear equations discussed, none of them from Spanish authors.

The works I refer to in this paper are written by the following Iberian authors: Marco Aurel (fl. 1552), Juan Pérez de Moya (c.1513-c.1597), Pedro Núñez (1502-1578) and Diego Pérez de Mesa (1563–c.1633). The criteria for selecting these authors are based on the relevance of their works, in the cases of Aurel and Núñez, his popularity, in the case of Pérez de Moya, and finally the step forward represented by the resolution of simultaneous linear equations in the unpublished manuscript by Pérez de Mesa.

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1. Leonardo de Pisa, better known as Fibonacci, was the autor of *Liber Abaci* (1202). This book contains the arithmetic and algebra that Fibonacci had accumulated during his travels. It is especially important because it introduced the Hindu-Arabic place-valued decimal system and the use of Arabic numerals into Europe.
2. See Franci, 1988, 7.
3. Kloyda, 1937.
4. Grcar, 2011, 169.

The aim of this article is to show the treatment given to the second unknown in these Spanish algebras of the second half of the 16th century. This analysis will provide new information about the algebraization process. Thus, from the first text analyzed, in which the procedure to solve problems that involved more than one unknown is rhetorical, to the last analyzed work in which the author sets up the system of equations and solves it by operating with equations. I also will advance an hypothesis about the main sources from which the authors drew their inspiration.[5]

Marco Aurel. The first algebra treatise in the iberian peninsula (1552)

The first book to be printed in the Iberian Península that could be regarded as an algebraic treatise is:[6] *Libro Primero de Arithmetica Algebratica,*[7] which was published in Valencia in 1552. Its author, Marco Aurel, was born in Germany and settled in Valencia, where he taught practical mathematics. This book consists of 24 chapters. In the six first chapters the author exposes the properties of whole numbers, fractions and proportions, the rule of three, currency exchanges and progressions. From the 7th to the 12th chapter, Marco Aurel deals with square roots and cube roots, binomials and apotomes and their square and cube roots. In the 13th chapter he states the characters that he will use afterwards. From the 14th to the 21st, and in the 23rd and 24th chapters, he deals with the rules to solve the different types of equalities. In the 22nd he addresses the binomials and apotomes, which he had already dealt with, by using the rule of thing.

In Chapter XVI of his algebra, Aurel addresses the systems of equations thus: "It deals with the rule of quantity, with some rules and requirements by which they are done, also known as the rule of the second thing".[8]

In this chapter the author solves eight problems with quite different wordings. The method to solve these problems consists in putting the second unknown in terms of the first one, the same for the third unknown, and so on. x being the first unknown, he puts *q* for

5. We must first take into account that Spain's communication with the rest of Europe almost ceased to flow from 1557, when groups of protesters were arrested in Seville and Valladolid. One year later, King Philip II presided over the first of a series of autos-da-fe that culminated in the burning of the Spanish Protestants. This ideological repression was a strict control of intellectual activity by royal power and the Inquisition, limited in principle to theology, but soon spreading to other fields. (for further information, see López Piñero, 1979). These circumstances made the availability of European reference works much more difficult in Spain.

6. This was the first algebraic treatise to appear in print, but not the first treatise in the Iberian Peninsula containing algebra, as Docampo has shown (Docampo, 2008).

7. The complete reference of this text is: *Libro Primero, de Arithmetica Algebratica, en el qual se contiene el arte Mercantivol, con otras muchas Reglas del arte menor, y lanRegla del Algebra, vulgarmente llamada Arte Mayor, o Regla de la cosa: sin la qual no se podra entender el decimo de Euclides, ni otros muchos primores, assi en Arithmetica como en Geometria: compuesto, ordenado, y hecho imprimir por Marco Aurel, nautral Aleman: Intitulado, Despertador de ingenios.*

8. *Trata de la regla de la quantidad, con algunas reglas, y demandas que por ella se hazen, que por otro nombre se puede llamar, regla de la segunda cosa* (Aurel, 1552, 108r).


LIBRO PRIMERO, DE ARITHMETICA Algebratica, enel qual ſe contiene el arte Mercantiuol, con otras muchas Reglas del arte menor, y la Regla del Algebra, vulgarmente llamada Arte mayor, o Regla de la coſa: ſin la qual no ſe podra entender el decimo de Euclides, ni otros muchos primores, aſsi en Arithmetica como en Geometria: compueſto, ordenado, y hecho Imprimir por Marco Aurel, natural Aleman: Intitulado, Deſpertador de ingenios. Va dirigido al muy magnifico ſeñor moſſen Bernardo Cimon, Ciudadano dela muy inſigne y coronada Ciudad de Valencia.

Con Priuilegio de ſu Mageſtad, por tiempo de diez años.

EN VALENCIA,
En caſa de Ioan de Mey, Flandro.
Año. 1552.

Figure 1. The cover of Aurel's *Algebra.*

the second unknown, and when the second is expressed in terms of the first he also puts *q* for the third, and so on.

In the first problem, the expression 1x+14Q[9] has to be divided in two parts according to certain conditions.

The second problem concerns three companions who want to buy a horse and each of them individually has not got enough money.

The third one[10] is about four people who owe money and we need to know how much money each of them has when the sum of the debts of every group of three is known. The fourth problem is the same as the third but is solved in a different way.

The fifth concerns three people who want to buy a vineyard and none of them has enough money. The way to afford it is by asking the others for a part of the purchase price.

The sixth has an over-elaborate wording. It is about three people who have a book, a scarf and a pair of gloves. They are given six stones and each of them has to take one, two, or three stones, but we do not know how many stones each person takes. Afterwards, 20

9. Q is the symbol that Marco Aurel links to the independent term. Two of the authors studied in this paper, Marco Aurel and Pérez de Moya link a symbol to the independent term. But this is not the case for Núñez and Pérez de Mesa.

10. There is a small mistake in the text and this problem is numbered with a 2 equal to the second one.

more stones are put on the table. One of the people has to take the same number of stones as before, another one double and the third four times the number of stones he took before. If the person who has the book takes as many stones as before, the one with the scarf takes double, and the one with the gloves four times as many stones, and knowing how many stones are left on the table, we have to work out how many each person took at the beginning.[11] In order to solve this problem, the author puts the number of stones that each person took in terms of the *x* quantity assigned to the first one, and also uses an auxiliary unknown *q,* which is assigned to the second one. Finally, he has to give *x* some values until he finds a consistent solution. The seventh problem is the same as the sixth, but the author solves it in a shorter way.

The eighth deals with four people who have money. Using information about how quantities would change if some of them gave money to the others and about the total amount, one has to find out how much money each person has.

Let us consider the second problem:

> *Three friends have a certain amount of money and they want to buy a horse whose price is 34 ducats. The first friend tells the others to give him the half of money they have, and with the money he has he will buy the horse. The second tells the others that if they give him a third of the money they have, he could also buy the horse. Finally, the third friend wants the others to give him a quarter of the money they have, and by adding his own money he will buy the horse.*[12]

This problem is a classic one of several men buying a horse,[13] where each man needs a share of the other's money to buy the horse. Several of these kinds of problems can be found in the 2nd part, distinctio 9, treatise 9 of the Pacioli's *Summa* (1494). Although Pacioli has used a second unknown in two different occasions, he introduces it in distinctio 8, treatise 6 under the heading: *quantum essential notandum*, for solving a problem of two numbers, when the sum of their squares and their product are known. He calls the unknowns *surd quantities*[14] and he proposes to call the first *co.* and the second *1 quantita.*[15]

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11. Aurel, 1552, 110r-110v.

12. Tres compañeros tienen dineros, y quieren comprar vn cauallo por 34 ducados. Dize el primero a los otros dos que le den la $\frac{1}{2}$ de lo que tienen, y con lo q̃ el tiene podra pagar el cauallo: el segundo demanda a los otros dos, que le den $\frac{1}{3}$ de lo que tienen, y con lo que el tiene tambien podra pagar el cauallo: el tercero demanda a los otros dos el $\frac{1}{4}$, y que podra justamente pagar el cauallo (Aurel, 1552, 108v).

13. A comprehensive history of the problem is given by Tropfke (1980, 608).

14. The expression *surd quantity* is used by other authors to refer to irrational quantities.

15. It is also the first problem that Cardano (1501-1576) solves in Chapter IX of his *Ars Magna* (1545). In this case, Cardano speaks about the quantity of money that the three men have. Indeed, it is also question IIII in the first book of the algebra of Pelletier (Pelletier, 1554, 107), with a similar wording to Cardano's.

The way that Aurel solves the second problem is similar to how Pacioli[16] solved the first problem of the 2nd part, distinction 9, treatise 9 of his *Summa.*[17] The procedure is also the same that used Rudolff to solve the problems in the “regula quantitatis” chapter, and in this case we find a problem with similar wording.[18]

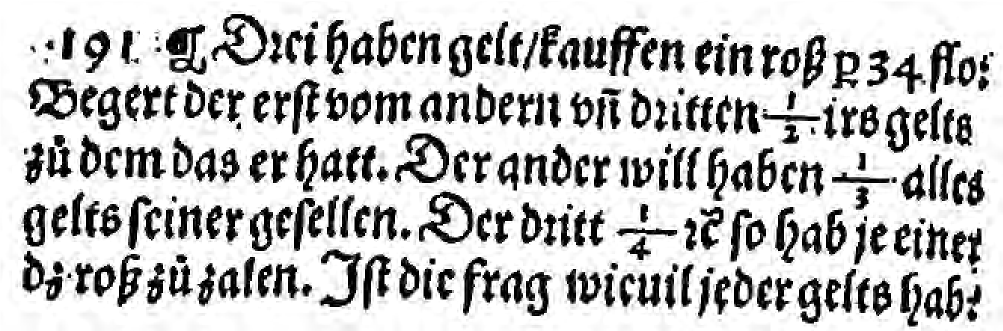
·:191·: ¶ Drei haben gelt/kauffen ein roß p 34 flo:
Begert der erst vom andern vñ dritten $\frac{1}{2}$ irs gelts
zů dem das er hatt. Der ander will haben $\frac{1}{3}$ alles
gelts seiner gesellen. Der dritt $\frac{1}{4}$ rc so hab je einer
dz roß zů zalen. Ist die frag wieuil jeder gelts hab?

Figure 2. The wording of a “men buying a horse” problem from Rudolff.

Let us now analyse the resolution of the second problem of Aurel, step by step. If we translate the wording of the problem into current algebraic language, we obtain:

$$\begin{cases} x + \frac{1}{2}\,y + \frac{1}{2}\,z = 34 \\ \frac{1}{3}\,x + y + \frac{1}{3}\,z = 34 \\ \frac{1}{4}\,x + \frac{1}{4}\,y + z = 34 \end{cases}$$

Aurel solves the problem in the following way: let us assume the first of the companions has *x* ducats. So, he needs *34Q-1x* to be able to buy the horse, a quantity that has to be equal to half of the other two.[19] Therefore, the other two have *68Q-2x* ducats, and altogether

16. Although the work of Pacioli was a reference for the authors from this period, the direct influence for Aurel was mainly Rudolff’s algebra, especially when using the second unknown, as well as the notation and the concrete wording of problems (See Romero, Massa, forthcoming).

17. The wording ot this problem is: three (men) have d (denars). The first says to the other two: if you give me half of your sum, I will have 90. The second says to the other two: If you give me a third of your sum, I will have 84. The third says to the other two: If you give me a quarter of your sum and plus 6, I will have 87. I demand the quantity that every man has.

18. Rudolff, 1525, 221.

19. Aurel is working with the expression that would correspond to a first equation in the system.

the three of them have *68Q-1x*. If the second one has *q* ducats, then the first and the third together have *68Q-1x-1q* ducats. A third of this quantity is, $22\frac{2}{3}Q-\frac{1}{3}x-\frac{1}{3}q$ to which we have to add the quantity that the second one has, so that we get:[20] $22\frac{2}{3}Q-\frac{1}{3}x+\frac{2}{3}q$. Then Aurel makes this quantity equal to 34 ducats, which is the price of the horse, and with this expression the quantity of the second friend can be expressed in terms of the quantity of the first.

He then assumes that the third friend has *q* ducats; that is, he has used *q* to find the quantity that the second one has in terms of the first one's quantity. Then he uses the same unknown again to indicate the money that the third one has. As before, the first and the second friend together have *68Q-1x-1q* ducats. One quarter of this quantity is $17Q-\frac{1}{4}x-\frac{1}{4}q$,[21] to which he adds the quantity *q* that the third one has and obtains $17Q-\frac{1}{4}x+\frac{3}{4}q$, which must be equal to the price of the horse, that is, 34 ducats. From this equality he gets the expression of the third friend's quantity in terms of the first one: $\frac{1}{3}x + 22\frac{2}{3}Q$. When he joins the three quantities together, he obtains: $1x+\frac{1}{2}x+17Q+\frac{1}{3}x+22\frac{2}{3}Q=1\frac{5}{6}x+39\frac{2}{3}Q$, which must be equal to *68Q-1x*, from which he gets *x=10*, and *22* and *26* for the second and third one's quantities of ducats.

Thus, with Aurel's procedure it is not possible to work with all the conditions at the same time when solving problems in which more than one unknown are involved.

Juan Pérez de Moya. The most popular algebra treatise on the iberian peninsula

Aurel's book seems to have been one of the main sources for the most popular algebra book written in the Iberian Peninsula in the XVI century, *Arithmetica practica y speculativa*[22] (Salamanca, 1562) by Juan Pérez de Moya (Santisteban del Puerto, ca. 1513 - Granada, ca. 1597), which went into about 30 editions. Pérez de Moya studied in Salamanca and he was a priest in his home village.

The *Arithmetica* consist of nine books. In the three first books, the author exposes the operations with whole numbers and fractions, the rule of the three and proportional distribution of benefits. The fourth is about practical geometry and the fifth about theorical arithmetic. In the sixth book, Pérez de Moya deals with the currency exchanges, and the seventh is devoted to the rule of thing. The eighth is about ancient currencies and mobile holidays. The last book is very different from the others and is written as a dialogue about arithmetic between two students, to convince the reader about the importance of this discipline.

20. Now Aurel is working with the expression that would correspond to the second equation of the system.

21. Aurel is now working with the expression that would correspond to the third equation in the system.

22. Fifteen Chapters of this book were published previously in his work *Compendio de la Regla de la Cosa o Arte Mayor* (1558). More information about the Arte Mayor in Spanish texts in 16th century can be found in (Massa, 2012).

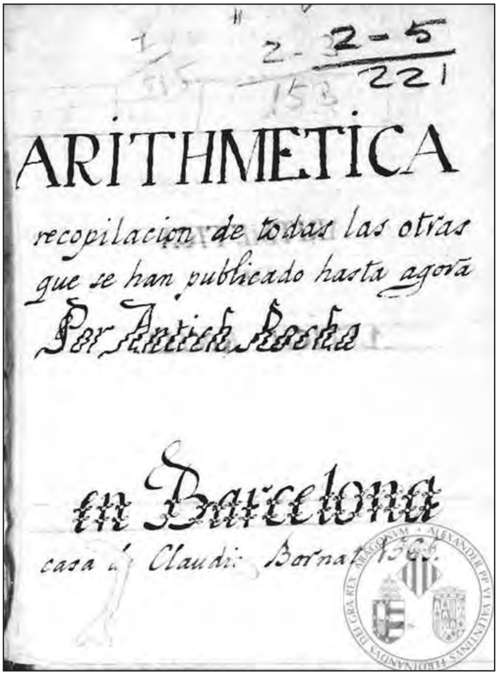

Figure 3. The cover of Antic Roca's *Arithmetica*

The *rule of quantity* is the 9th article of the 13th chapter of the 7th book, and in it the author solves only three problems, the first of which is similar to Aurel's first problem and is resolved with the same procedure. A quantity has to be divided in two parts according to certain conditions.[23]

The other two examples are problems that we would solve nowadays by a three equation system with three unknowns, and these problems are formulated with numbers; they have no concrete context dealing only with relations between numbers.

The wording of the second one is as follows:

> *Give me 3 numbers such that by adding the first number and the second one with half of the third the sum equals 30, and the second number with the third added to a third of the first number equals 30, and the third number with the first added to a quarter of the second number equals 30. It is required that ..., etc.*[24] (Pérez de Moya, 1562, 601-602).

23. Pérez de Moya, 1562, 600-601.

24. *Dame 3. numeros de tal condicion, que summando el primero y el segundo con la mitad del tercero la summa sea 30. y el segundo y tercero con el tercio del primero hagan 30. y el tercero y primero con el quarto del segundo hagan 30. Demando ¿&c.*

Pérez de Moya solves this problem in a similar way to Aurel, with an auxiliary unknown assigned to both the second quantity and the third. Unlike Aurel, he does not assign an *x* to the first quantity but rather *co.* and neither does he use the signs “+” and “-“ to indicate addition and subtraction, but rather “p” and “m” , and the sign that Pérez de Moya links with the independent term is *n.*[25] However, he does use *q* to indicate the second and third quantities.

Regarding the resolution of these problems, it is clear that the author follows Aurel, although he does not quote him as a source.

In a later edition, published in 1573,[26] the author extended his work, particularly in the chapter devoted to the second quantity. In the preface addressed to the reader, Pérez de Moya states that there are improvements to be made in his works, and that this latest edition is a compilation of what he had written in vulgar language and also what he had written in a booklet in Latin. In his own words:

> *Therefore, recognizing that in my works there were things which required revision, I agreed to provide such amendments as seemed necessary: by amending myself many things from my works printed so far and adding to each subject what I thought should be made known to improve it. And thus this book appears as a sum of what has been written in the vernacular, and the best and most important things that we included in a booklet in Latin entitled* Sylua*, with the addition of more than two hundred written pages.*[27]

In this new edition, the rule of quantity also appears in the seventh book, but in this case it is the sixth article of the 51st chapter. Six problems are solved, and with important differences from the previous edition. In the 1562 edition, the fractions were expressed in a rhetorical way and the only fraction written explicitly was numerical, at the end of the first problem. In the 1573 edition, fractions that contain unknowns are explicitly expressed. Another more important novelty is the assignation of a letter to the third unknown, different

The author left the question unfinished, probably taking for granted that the reader knew how this sentence ended. For the resolution of this problem, on the eighth line of page 602 there is a slight error: where it says “the other 2 will be 30.n.p.media.co.”, it should say “the three will be 30.n.p.media.co.”.

25. Another author that should be taken into account is Antic Roca , whose *Arithmetica* is written along the lines of Marco Aurel and Pérez de Moya . He deals with the Major Art in the fourth book, but there are no references to the rule of quantity. Although this author makes no reference to the *second quantity,* his algebra is interesting for a panoramic view of algebraic Spanish works in the 16th century. For further information, see (Massa, 2008).

26. The work was written at least two years before, since the King’s permission to print it is dated 9th December 1571.

27. *Por tanto, conociendo que en mis obras avia cosas que requerian censura, acorde proveer a lo que me pudieran emendar: emendando yo mismo muchas cosas de mis obras hasta agora impressas, y con mejoria añadido sobre cada materia lo que me pareció que bastava saberse. Y assi va agora este libro como una summa de lo que se ha hecho en lengua vulgar, y lo mejor y mas importante de las cosas naturales que pusimos en un librillo de Latin intitulado* Sylua*, y añadidos sobre todo mas de docientos pliegos de escriptura (*Pérez de Moya, 1573, introduction without pagination).

TRATADO DE MATHEMATICAS EN QVE SE CONTIENEN COSAS DE ARITHMEtica, Geometria, Cosmographia, y Philosophia natural. Con otras varias materias necessarias a todas artes Liberales, y Mechanicas. Puestas por la orden q̃ a la buelta de la hoja veras.

Ordenado por el Bachiller Juan Perez de Moya, natural de Sant Esteuan del Puerto.

DIRIGIDO A LA S. C. R. M. DE DON Phelipe Rey de España nuestro señor.

Con licencia, y priuilegio Real de Castilla y Aragon.

EN ALCALA DE HENARES.

Por Iuan Gracian. Año de 1573.

Figure 4. The cover of Pérez de Moya's *Tratado de Mathematicas.*

from the letter assigned to the second one. Although he still refers to the second unknown as "quantity", he does not assign to it the letter *q* but rather the letter *a*. To the other unknowns, the author assigns the letters *b, c, d,* and so on.

The first (p. 597) and the second problem (p. 598) solved by Pérez de Moya are numerical and very similar to each other. The translation of their wording into current algebraic language would be as follows:

$\begin{cases} 6x + 2 = 7y - 14 \\ xy = 90 \end{cases}$ for the first problem, and $\begin{cases} 4x + 16 = 3y + 10 \\ xy = 60 \end{cases}$ for the second one.

In both cases, the aim is to solve what are currently known as nonlinear systems, which the author does by isolating the second unknown from the first equation, replacing the second one and solving the obtained quadratic equation.

The third problem (p. 598) is also about relations among numbers: the goal is to find two numbers whose sum is 12, the quotient between the biggest and the smallest being 17, which at present we would translate into symbolic language like this: $\begin{cases} x + y = 12 \\ \dfrac{x}{y} = 17. \end{cases}$

Pérez de Moya solves this problem in different ways. First of all, he isolates the second unknown, from the expression that corresponds to the one I have written in the system as the first equation. He then replaces it in the second one and solves the obtained equation. Let us see how the author expresses the replacement:

> *Divide the thing (which is one number) by twelve numbers minus one thing (the value of the quantity that you put for the other one), and we put the partition above the ___ with a line between them as a fraction, like this* $\frac{1co.}{12n.m.1co.}$ *(as shown in Chapter 29), which you make equal to 17n. as you wanted. Then reduce this equality to whole numbers, multiplying twelve n. m .1 co. by 17 (as stated in the fourth article of Chapter 44), and make the result equal to one co. (which is the numerator of this fraction), and you will obtain the equality 204 n. m. 17 co. yg. to one co. Take 17 co. away (which has a minus sign in one part) and add them to the thing (which is in the other part) and 204 n yg. to 18 co. will remain. This means that 18 things are equal to 204 numbers.*[28]

The second way to solve the problem consists in assuming that the biggest term is the second one, so when we make the replacement in the expression that we have written in the second equation, the division will be $\frac{12n.m.1co.}{1co.}$ that in modern notation would be $\frac{12-x}{x}$. He later considers other ways that would correspond to the isolation of the unknowns in the second equation and the replacement in the first equation. In these cases, Pérez de Moya also considers different possibilities for finding the other unknown.

The fourth (p. 601) and fifth (p. 603) problems are similar and both of them have a concrete context. They concern three people who have money and the part they need from each other to obtain a specific quantity. In the fourth problem the relations that are expressed refer only to two of the people. However, in the fifth problem each relation includes the three people. In these cases, Pérez de Moya does not speak about quantity when referring to the second unknown. He assumes that the first person has "one thing of *reals*",[29] the second "one *a*" and the third "one *b*". In both problems he works with the relations between quan-

28. *Parte una cosa (que es el un numero) por doze numeros, menos una cosa (que es el valor de la quantidad que pusiste por el otro) poniendo la partición sobre el partidor con una raya en medio como quebrado, deste modo* $\frac{1co.}{12n.m.1co.}$ *(como se mostro en el capitulo 29) lo qual ygualaras a los 17n. que quisieras que fuera. luego reduze esta ygualacion a enteros, multiplicando doze n. m .1 co. por 17 (como manda el quarto articulo del cap. 44) y lo que montare ygualalo a la una co. (q es numerador deste quebrado) y quedara la ygualacion 204 n. m. 17 co. yg. a una co. quita la 17 co. (que vienen menos en la una parte) y juntalas con la cosa (que viene en la otra) y quedaran 204 n yg. a 18 co. quiere decir, que 18 cosas, son tanto o valen tanto, como 204 numeros* (Pérez de Moya, 1573, 599).

29. A real was an unit of currency at that time.

tities and sets the second and third unknowns in terms of the first one. Then when he has an equation with only one unknown, he solves it and then finds the unknowns with the relations he has obtained. When he solves the fifth problem, he makes an interesting generalization that is important to point out, with the aim to explain how to solve systems of linear equations, whatever the number of unknowns they have, that is, a general procedure:

> *And if we had four companions, you would proceed in this way: We would put for the fourth a quantity under this letter* c. *and we would do with this what has been done with* a. *and* b.*, and this way we can proceed infinitely.*[30]

The sixth and last problem in the 1573 edition is the first one in the 1562 edition:

> *From two and two thirds thing, plus 18, make two parts so that when taking twelve of the second part and adding it to the first one, the first one is the triple of what is left on the second one, and plus three.*[31]

In this case, the result cannot be given in numbers because it depends on the value of the unknown in the wording.[32]

In both editions, Pérez de Moya checks the solution by giving the unknown the value of 6, although he states that it can have any value. In the 1573 edition he adds this paragraph at the end:

> *From which it can be seen that, without knowing the value of the thing, these requirements can be met and putting as we see fit the value that we wish.*[33]

Although Pérez de Moya's quotes Cardano and Núñez in this edition,[34] he follows none of these authors in their treatments of the second unknown.

30. *Y desta manera procederas, si fueran quatro compañeros, poniendo por el quarto una quantidad debaxo desta letra c. y haziendo con ella lo que se hizo con la a.y la b. y asi se puede proceder en infinito* (Pérez de Moya, 1573, 604).

31. *Haz de dos y dos tercios cosa, mas 18 numeros, tales dos partes, que quitando doze de la segunda parte, y añadiendola a la primera, sea la dicha primera el triplo de lo que quedare a la segunda, y mas tres* (Pérez de Moya, 1573, 604).

32. The difference between the ways of solving this problem in the two editions is not the procedure itself, which is the same, but the fact that in the 1573 edition there are more explanations, and fractions are expressed with the current notation rather than in the rhetorical way used in the first edition.

33. *De lo qual se sigue, que sin saber el valor de la cosa, se pueden hazer estas demandas, poniendole a nuestra volutad el valor que nos pareciere* (Pérez de Moya, 1573, 605).

34. *Otras varias, y diversas ygualaciones ay que dexo de poner, porque para sabios no son menester, y para principiantes no se entenderan. Quien quisiere ver algo, lea el decimo d Arithmetica de Cardano. Y en las ygualaciones que el cubo y cosa, se ygyualaren a numero. Lea al doctor Pedro Nuñez, al fin del tratado de Algebra que lo trata mas discretamente, que ninguno de los que antes del lo inventaron* (Pérez de Moya, 1573, 589).

Pérez de Moya’s work contains an important step forward as opposed to Aurel’s work, thanks to the fact that he kept the symbol *co.* for the first unknown and he assigned the letters *a, b, c,* etc. to the following unknowns. This type of assignation, which gives relevance to the first unknown, is also highlighted in the Stifel’s[35] comment of *Die Coss de Rudolff*.

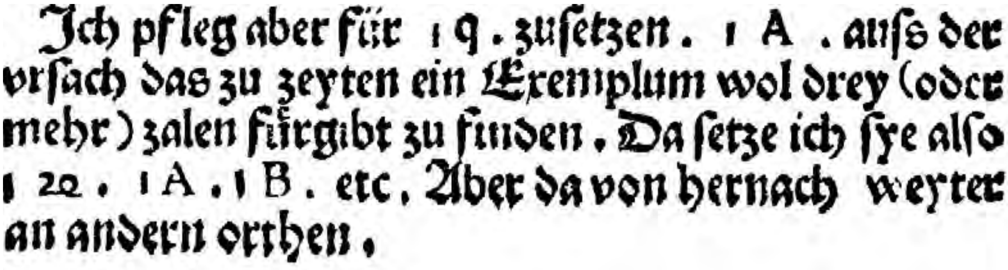
Ich pfleg aber für 1 q. zuſetzen. 1 A. auſs der vrſach das zu zeyten ein Exemplum wol drey (oder mehr) zalen fürgibt zu finden. Da ſetze ich ſye alſo 1 2e. 1 A. 1 B. etc. Aber da von hernach weyter an andern orthen.

Figure 5. Notation of the unknowns in the comment of *Die Coss* of Rudolff by Stifel

Here Stifel states that he prefers to use *1A* instead of *1q* because sometimes there are examples with three or more numbers, and we will use *1A, 1B*, etc.[36] However, there is no evidence that Pérez de Moya was inspired by Stifel’s comment in *Die Coss.*

The assignation of the same letter to two or more unknowns makes it impossible to work with all the unknowns at the same time, preventing the system from being formulated explicitly.

Thus, as regards the second quantity, one may see the development of algebra in the work of Pérez de Moya. Assigning different values to different unknowns constitutes an important step, even though the method used does not lead him either to operate with the equations or to make the system explicitly.

Pedro Núñez (1567)

However, probably the most outstanding mathematician in the Iberian Peninsula during the XVI century was the Portuguese Pedro Núñez (Alcácer do Sal, 1502- Coimbra, 1578), who studied medicine and mathematics and worked as a mathematics lecturer at the universities of Salamanca and Coimbra. His book *Libro de algebra en arithmetica y geometria,* to which we will refer, was written in Portuguese during the 1530s, but was not published until 1567 and in Spanish.

35. In Heeffer, 2010, 77-82, the autor points out the innovation that Stifel introduces for the notation of the second and other unknowns.

36. Stifel, 1553, 186r.

It is a very long work, and different from Aurel's and Pérez de Moya's algebras. The interval between when it was written and its publication makes it difficult to analyze. There are many references to works after the 1530s that could not have been in the original version and which shaped the work.

In the 5th and 6th chapters of the third[37] main part, Núñez solves some problems that we would nowadays solve using systems of equations.

In the 5th chapter: *De la Practica de las Reglas de Algebra en los casos de Arithmetica,* Núñez solves 110 problems in which more than one unknown is involved. Some of them are linear and some others not. The resolution is rhetorical and the author makes no reference to a *rule of quantity*.

In the 6th chapter: *De la regla de la quantitat simple o absoluta*, Núñez solves three problems.[38] The author says that the rule of simple or absolute quantity is different from the others and that we use it in two ways. The first way is a substitution of the *rule of the thing* to perform the equality with the help of the term *quantity,* while the second is to do « position » over « position ». With this last expression, Núñez refers to problems whose wording contains one unknown, as we will see in the second problem of this chapter.

The first problem he poses is akin to the one we have given above as an example of Aurel's resolution, the classical problem about the three men who want to buy a horse but without any context.

Nuñez's wording is as follows:

> *We have three numbers, the third number with half of the second equals 32, the second number with a third of the other two equals 28, and the third number with a quarter of the other two equals 31. We want to know what each of them is.*[39]

In modern notation we could write:

$$\begin{cases} x + \dfrac{1}{2}\ (y+z) = 32 \\ y + \dfrac{1}{3}\ (x+z) = 28 \\ z + \dfrac{1}{4}\ (x+y) = 31 \end{cases}$$

37. This work is divided into three main parts. For further information about its structure, see (Bosmans, 1908, 4-19).

38. According to Bosmans, this chapter is one of the most interesting in the *Algebra* of Pedro Núñez (Bosmans, 1907, 168).

39. *Tenemos tres numeros, que el primero con la mitad de los otros, haze 32 y el segundo con el tercio de los otros dos, haze 28 y el tercero con el quarto delos otros, haze 31 y queremos saber quanto es cada uno dellos* (Núñez, 1567, 224v).

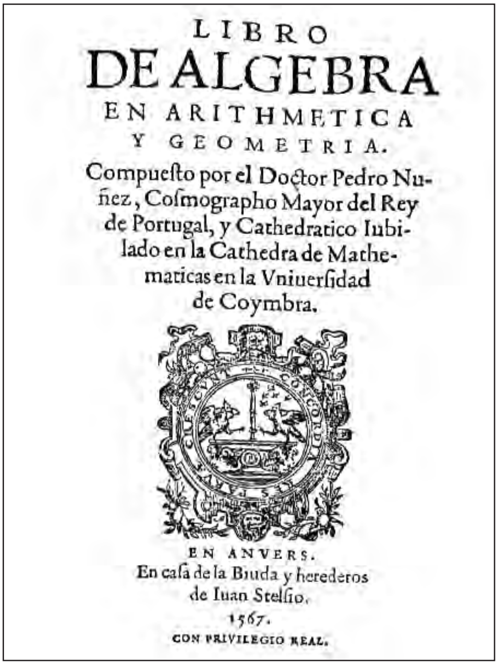
LIBRO
DE ALGEBRA
EN ARITHMETICA
Y GEOMETRIA.
Compuesto por el Doctor Pedro Nuñez, Cosmographo Mayor del Rey de Portugal, y Cathedratico Iubilado en la Cathedra de Mathematicas en la Vniuersidad de Coymbra.
EN ANVERS.
En casa de la Biuda y herederos de Iuan Stelsio.
1567.
CON PRIVILEGIO REAL.

Figure 6. The cover of Pedro Núñez's *Algebra.*

Núñez assumes that the first number is 1 *co*. Then, half of the second number and third will be[40] 32m̃1co. and this expression doubled, that is 64m̃2co., will be the second and third number together. As the first number is 1 *co*., all three numbers will be 64m̃1co. Here Núñez is working with what I set as a first equation of the system, and he has expressed the sum of the three numbers in terms of the first one. After that, he assumes that the second number is "1 quantity", and from what he obtained with a reasoning based on the second equation, he arrives at the *quantity* in terms of the *thing* and also the third number in terms of the *thing*. It is important to point out that Núñez uses no abbreviation for the *quantity,* and that he gives no name to the third unknown. Núñez's reasoning is as follows: As the three numbers together make 64m̃1co., the first and the third together will make 64m̃1co.m̃1 quantitat. If we take the third part of this expression and we add to it 1 quantity, which is the value of the second number, we obtain $21\frac{1}{3}$ *m̃.* $\frac{1}{3}$ co.p̃ $\frac{2}{3}$ of quantity, from where he obtains for the value of 1 quantity 10p̃$\frac{1}{2}$ co. and for the value of the third one[41] 54m̃ 2co.$\frac{1}{2}$. Once he has expressed the three numbers in terms of the first one, he works with the rela-

40. m̃ is the symbol that Núñez uses to indicate the subtraction, and p̃ is the one he uses to indicate the addition.

41. I would like to remark that to find the expression of the third number in terms of the first, Núñez gives no name to the third number.

tion that we have written as the third equation of the system, and obtains the values 12, 16 and 24 for the first, second and third numbers, respectively.[42]

The second problem he addresses in this chapter is an example of what he calls "position" over "position", the second way of using the absolute quantity. After solving this problem, he gives an alternative method without using the quantity, but rather the proportion theory from Euclid's *Elements*.

The problem is as follows:

> *Let us divide* 1 co. p̃.3. *into two parts so that when adding 4 to the first number and subtracting 5 from the second, the first number would be six times more than the second.*[43]

In order to solve this problem, he assumes that the first part is 1 *quantity* and therefore the second one will be 1 co. p̃.3.m̃.1 *quantity*. Next, he adds 4 to the first part and takes 5 from the second one. Since the result of the first part is six times more than the result of the second one, he concludes that 6co.m̃.6 quantitats m̃.12. must be equal to 1 quantitat p̃.4., and from this he arrives at the value of the *quantity*, which is the first part, as well as the value of the second part.

He then solves the problem in another way without using the *quantity.* He says that if we add 4 to the first part and subtract 5 from the second one, then one unit of the sum will be lost and thus the sum will be 1 co.p̃.2. As the ratio of the result of the first part to a result of the second part is the same ratio of 6 to 1, the ratio of the total amount to the second part will be the ratio of 7 to 1,[44] and by the rule of three we can determine this second part:

If 7 gives us 1, how much will we obtain with 1co.p̃.2? The author says that the second one should be multiplied by the third and the result should be divided by 7, obtaining $\frac{1}{7}$co.p̃.$\frac{2}{7}$, which is the result of the second part when we subtract 5 units from it, so the second part will be $\frac{1}{7}$co.p̃.5$\frac{2}{7}$. On subtracting this quantity from the total amount, we obtain the first part.

42. Núñez had already solved this problem in the previous chapter without using the "quantity", but rather using the relations between the quantities involved, and in a clever way. This is problem 51 in the 5th chapter. At the beginning, the way to solve it is the same one we use in the following chapter: Núñez assumes that the first number is *1 co*. Then half of the second plus the third together make 32m̃1co. and two times this expression, that is 1co., will be the second and the third altogether. Then he takes into account the relation between the quantities that we have written as the second equation in the system, in order to say that if one third of the first, that is $\frac{1}{3}$ co. is taken from 28, we obtain the value of the second number and a third of the third number. On subtracting this quantity from 64m̃2co., that is, the value of the 2nd and the 3rd, we are left with 36m̃1co. $\frac{2}{3}$, which is the value of $\frac{2}{3}$ from the third number. If we add this quantity to its half, we obtain the value of the third number: 54m̃2co.$\frac{1}{2}$, If we subtract this quantity from 64m̃2co., then we are left with 10p̃$\frac{1}{2}$ co. He then continues as in the following chapter.

43. *Partamos* 1 co.p̃.3. *en tales dos partes, que dando ala primera 4 y sacando dela segunda 5 resulte la primera seis veces mas que la segunda* (Núñez, 1567, 225v).

44. Here Núñes refers to a "joint ratio" that Euclides proved in the 5th book of his *Elements,* but he does not quote the proposition.

The third and last problem leads to the solving of a quadratic equation. Núñez says that Cardano solved this problem by *the* "rule of quantity", but it can be solved in an easier way by *the* "rule of the thing". In fact, the first problem from Chapter X in Cardano's Ars Magna is as follows:

> *Inuenias duos numeros, quorum quadrata iuncta, sint 100 & productum unius i alterum duplumsit aggregato eorum.*

This he solves from the rules that he had previously demonstrated, and by using the properties of proportions.

Núñez says:

> *Let us find two numbers, the sum of whose squares is 100 and whose product is twice the sum of both.*[45]

This solution is more elegant than that of Cardano's. Nuñez's skill lies in the choice of the unknown. Let us assume that the two numbers together are 1 *co.*, so its square will be *1 ce*. Then he uses the equality that we would express as: $(x+y)^2 = x^2 + y^2 + 2xy$,[46] and concludes that 100p̃.4co. equals *1 ce*. Then by applying the rules of compound conjugations, he finds the value of the thing, that is, the sum of the two numbers, 2.p̃.R.104.[47] Therefore, he also knows their product, which is 4.p̃.R.416. The author then says that it is necessary to divide 2.p̃.R.104 into two parts so that the product makes 4.p̃.R.416, and that is why the rule of proportional means or algebra can be used. Núñez assumes that one of these parts is *1 cosa,* so the other will be 2.p̃.R.104m̃.1co. By multiplying these two parts and making their product equal to 4.p̃.R.416, he obtains the solution and then checks it.

Núñez's sources are mainly Pacioli and Cardano, although he looks for alternative methods to the ones they used. As already mentioned above, Núñez appears somewhat sceptical about the use of this rule, which may be seen from some of his statements:

> *And in all cases where Fray Lucas works by the quantity, we work by the rule of the thing, without the help of this term quantity.*[48]

Further:

> *Ieronymo Cardano found many rules of quantity, by which he solves many questions which could have been solved with greater ease by the rule of thing.*[49]

45. *Busquemos dos numeros, cuyos quadrados juntos en una suma, sean 100. y que lo que se haza multiplicando vno por otro, sea duplo de la suma de entrambos juntos* (Núñez, 1567, 226v).

46. Núñez sets out this equality rhetorically by referring to the 4th proposition in the 2nd book of Euclid's *Elements*.

47. In current language $2+\sqrt{104}$.

48. *Y todos los casos que Fray Lucas practica por la quantidad, practicamos nos por las Reglas de la cosa, sin ayuda deste termino quantidad* (Núñez, 1567, 225v).

49. *Ieronymo Cardano hallo muchas Reglas de quantidad, por las quales resuelve muchas questiones que trae, podiendose muy bien resolver por las Reglas de la cosa, y con mas facilidad* (Núñez, 1567: 226v).

Diego Pérez de Mesa (1598)

The algebra of Pérez de Mesa forms the second part of Manuscript 2294 dated 1598, which can be found in the Library at the University of Salamanca. It is a double faced 100 page treatise entitled: *Libro y tratado del arismetica y arte mayor y algunas partes de astrología y matematicas compuestas por el eroyco y sapentisimo maestro El Licenciado Diego perez de mesa catredatico desta Real ciudad de Sevilla del año de 1598.*[50] The part dealing with algebra is named by the author *Tratado y Libro de arte mayor o algebra.*[51] It begins on page 60 and is composed of an introduction and 23 chapters. In the three first chapters, Pérez de Mesa deals with numbers and their properties. In the fourth, he exposes his idea of algebra and the fifth is about proportional numbers. The sixth to the tenth are about the whole numbers and their operations, and the eleventh to the sixteenth about fractions and their operations. The seventeenth is about rational and irrational numbers and in the last six the author deals with equations.

Pérez de Mesa solves the systems of equations[52] in the last chapter, which he entitles "On the Rule of Quantity". He says that "the writers"[53] talk about the rule of quantity when they cannot arrive at a solution with only one unknown, and that to solve the situations involved, not all the writers use the same method". He adds that since assigning the same symbol to every unknown might give rise to confusion, the name of "thing" is usually given to the first unknown, while the other unknowns are called "quantities"[54] and are given different names. For the second unknown these authors put "a", for the third, "b", for the fourth, "c", and so on. Sometimes none of them is assigned the name "thing", and the first is called "a", the second "b", the third, "c", and so on.

Pérez de Mesa's way of solving the systems of equations is different from that of other Spanish authors. Rather than taking an auxiliary unknown, he puts « a » for the first unknown, « b » for the second, and so on. To solve a system, Pérez de Mesa reduces the expressions until he obtains a single expression that consists of one unknown on one side and a number on the other side. Then by applying the "simple canon[55]" he obtains the value of the unknown. After obtaining the first value, he obtains the others by replacing the value obtained in the other equations. This is the method that now is known as back-substitution.

The first exercise that he solves is as follows:

50. Book and treatise on arithmetics and great art and some parts of astrology and mathematics written by the heroic and very wise master, the graduate Diego Pérez de Mesa, professor of this royal city of Sevilla in the year 1598.

51. Treatise and book of great art or algebra.

52. For a more detailed study of the work of Pérez de Mesa related to equations and the treatment of the *second quantity*, see (Romero, 2008).

53. Pérez de Mesa refers to something that is done for the "authors" or to the "writers", to speak about some rules that are assumed to be well known.

54. Pérez de Mesa, 1598, 99.

55. Pérez de Mesa calls "canons" the rules for solving equations.

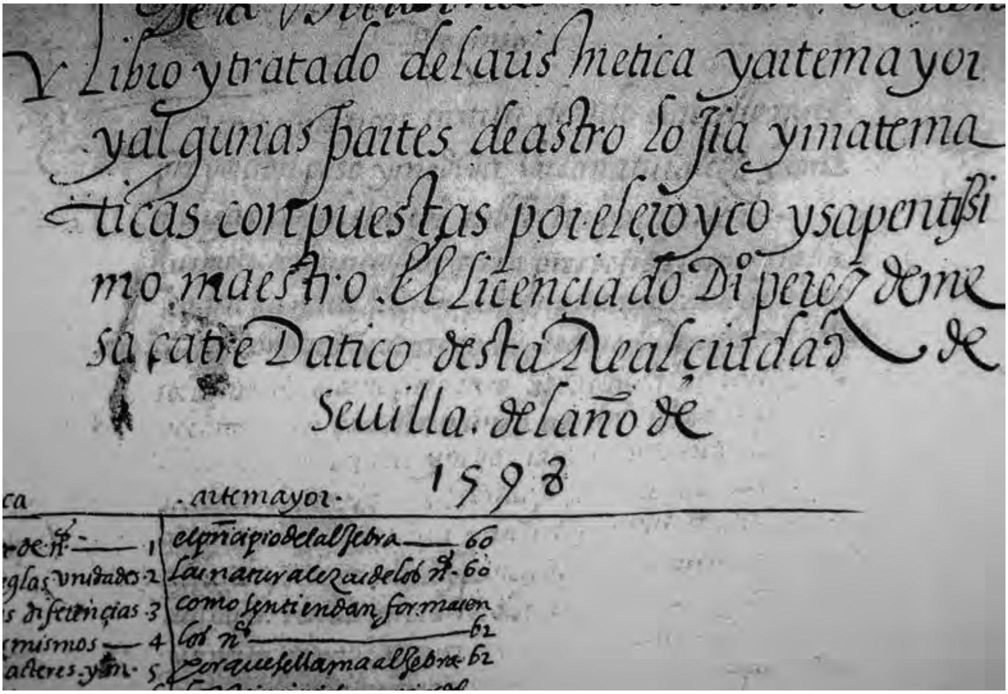

Figure 7. The title in the first page of manuscript 2294 by Diego Pérez de Mesa.

> *Two numbers are given, so half of the lower with the higher is 15 and the lower with a third of the higher is 10*[56].

Pérez de Mesa calls the larger number « a » and the smaller « b ». He says that the fractions should first be reduced to whole numbers and then obtains the equation that would nowadays be written: 2a+b=30 and a+3b=30. He solves the system by using the reduction method and by multiplying the second equation by 2 and subtracting it from the first equation, thereby obtaining 5b=30, and thus the value 6 for « b ». He replaces this value in the first equation and finds 12 for « a ». Then he sets out the system as follows.

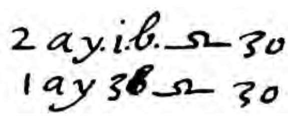

Figure 8. System of equations with 2 unknowns set up in Pérez de Mesa’s manuscript.

56. Dénse dos n^{os} que sumando la mitad del menor con el mayor hagan 15 y el menor con el tercio del mayor haga diez (Pérez de Mesa, 1598, 99).

Although he solves this problem in a rhetorical way, I would like to point out the symbols he employs. He uses the letter “y” for the “plus sign”, as he did when operating with polynomial expressions, and puts **Ω** for the equality. This is the first time in his manuscript that he uses a symbol to indicate the equality. We would do well to remember that neither Bombelli (1572) nor Stevin (1585) nor Viète (1590) used any symbol[57] to indicate the equality.

Let us see the solution to the second problem:

Three numbers are given such that the largest together with a third of the other two make 17; the second number with a third of the others make 15, and finally the last number with a third of the others make 13.

First, he removes the denominators and he writes the system as follows.

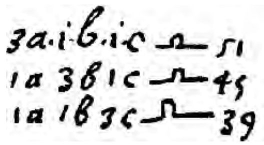

Figure 9. System of equations with 3 unknowns set up in Pérez de Mesa’s manuscript.

In modern notation:

$$\begin{cases} 3a + b + c = 51 \\ 1a + 3b + c = 45 \\ 1a + 1b + 3c = 39 \end{cases}$$

Then he multiplies the second equation by 3 and subtracts it from the first. After that, he multiplies the third equation by 3 and also subtracts it from the first, thereby obtaining the system that we write as follows:

$$\begin{cases} 8b + 2c = 84 \\ 2b + 8c = 66 \end{cases}$$

He also solves this system by what is now known as the reduction method and obtains 6 for the value of « c ». Then, by substitution in the first equation, he obtains 9 for the value of « b ». Finally he puts the values obtained in the equation: 3a+b+c=51 and obtains the value 12 for « a ».

The author tackles the last problem that leads to what we now call a nonlinear system and solves it by the method of substitution.

When referring to the solution of the system of equations, Pérez de Mesa is probably taking Buteo[58] as his source, since on the first page of his manuscript devoted to alegebra he

57. For further information about the evolution of algebraic symbolism, see (Cajori, 1993).

58. Joannes Buteo is the Latinized name of Jean Borrel, a French mathematician (ca. 1492-ca. 1570).

quotes this author, first calling him “Triputeon”[59] and later “Puteon”. Buteo[60] makes the system of equations explicitly thus operating with them in the same way that Pérez de Mesa will subsequently do.

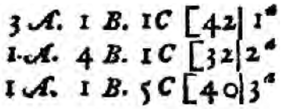

Figure 10. System of equations with 3 unknowns set up in Buteo’s *Logistica* (Buteo, 1559, 190).

We would also like to point out the fact that Pérez de Mesa considers all the unknowns in the same rank by assigning them different letters in alphabetical order.

Concluding remarks

The introduction of the second unknown was very important in the process of algebraization of mathematics, because it constituted a step forward in the development of the concept of an equation. The use of a second unknown contributed to the evolution of the idea of an equation, from a tool for solving some kinds of problems to the understanding of an equation as a new object of algebra with which one can operate, and thus also to the consideration of algebra as a discipline in its own right.

As we have shown, the stages in this process are reflected in the Spanish 16th century algebraic texts.[61] We can observe the different methods, starting with that by Aurel, in which the second unknown is called “quantity”, like the third one; Núñez’s algebra, a work by a somewhat sceptical author with the use of the second unknown, to Pérez de Moya’s work, published in 1573, in which different signs are assigned to different unknowns, although the first one retains a special name. Moreover, we have seen the furthest stage in the process in Pérez de Mesa’s manuscript, where the author writes the simultaneous equation system explicitly, treats the unknowns with equal rank and operates with the equations.

Although Núñez is quoted in the fourth chapter of Pérez de Mesa’s algebra, when refers to the names for the unknowns, and even though Pérez de Mesa follows Núñez in some aspects of his work, he is not followed by Pérez de Mesa in his skepticism about the use of the second unknown.

Thus, an important step referring to symbolic reasoning in the Iberian works studied is taken in the resolution of systems of equations in the algebra of Pérez de Mesa, when the method of reduction that involves operating with equations is used. It is probably not by chance that Pérez de Mesa introduces the sign Ω for the equality when he considers the equations, consciously or not, as a new object in the algebra.

59. Pérez de Mesa, 1598, 61.

60. Buteo, 1559, 160.

61. It is not easy to follow the thread of the second unknown throughout the different texts. A deep study on the early occurences of the second unknown in European texts can be found in (Heeffer, 2010).

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